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A RECURSIVE ALGORITHM FOR SUMS OF POWERS USING INTEGRATION

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Abstract: **In the attached article, the authors present a recursive algorithm that uses definite integration to generate formulas for the sum of positive integral powers of the first *n* positive integers.**

Keywords: **recursive algorithm, sums of powers, Integration.**

1. INTRODUCTION

Over the centuries, people from various parts of the world have established formulas to calculate the sum of positive integer powers of the first n positive integers. Some of the most noteworthy of these formulas are cataloged and discussed in [1], [2], [3], [4], and [5]. Blaise Pascal, Jacob Bernoulli, and Johann Faulhaber developed explicit polynomial formulas in the 17th century [3].

More recently, in April 2017, one of the authors attended a talk [6] at which formulas involving matrices, Stirling numbers, and Bernoulli numbers were presented for the sum of positive integer powers of the first n positive integers.

In this article, we introduce a recursive algorithm that employs definite integration to derive formulas for the sum of positive integer powers of the first n positive integers. The method presented here is recursive in the sense that for each $k = 0, 1, 2, \ldots$, the formula for the sum of the $(k+1)^{\text{st}}$ powers of the first n positive integers is obtained from the formula for the sum of the k^{th} powers of the first n positive integers.

2. THE ALGORITHM AND ITS PROOF

Algorithm: Let k be a fixed nonnegative integer, and suppose that

$1^k + 2^k + 3^k + \ldots + n^k = p_k(n)$, where n is a positive integer and $p_k(n)$ is a polynomial function of n. Then, to obtain a polynomial function of n, $p_{k+1}(n)$, such that $1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} = p_{k+1}(n)$, Step 1 is to compute the value of the definite integral $\int_0^n \left[(k+1)\cdot p_k(x) + C\right] dx = (k+1)\int_0^n p_k(x)\, dx + Cn$, where C is a constant. Step 2 is to determine the value of C by substituting 1 for n in the equation $1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} = (k+1)\int_0^n p_k(x)\, dx + Cn$. Then,

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$p_{k+1}(n) = (k+1)\int_0^n p_k(x)\ dx + Cn$, with C replaced by its value.

Note, in particular, that when $n = 1$, the equation

$1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} = (k+1)\int_0^n p_k(x)\ dx + Cn$ reduces to

$1 = (k+1)\int_0^1 p_k(x)\ dx + C$, so $(k+1)\int_0^1 p_k(x)\ dx = 1 - C$.

Before we prove the algorithm, we first state and prove the following lemma.

Lemma: $(k+1)\int_n^{n+1} p_k(x)\ dx = (n+1)^k - C$

Proof of Lemma: $(k+1)\int_n^{n+1} p_k(x)\ dx$

$$= (k+1)\int_0^n \left[p_k(x+1) - p_k(x)\right] dx + (k+1)\int_0^1 p_k(x)\ dx$$

$$= (k+1)\int_0^n (x+1)^k\ dx + (k+1)\int_0^1 p_k(x)\ dx$$

$$= (n+1)^{k+1} - 1 + (1 - C)$$

$$= (n+1)^{k+1} - C$$

Note that in the above proof, we used the fact that if two polynomials agree at all positive integers, then they agree at all real numbers, as well.

Proof of Algorithm: The proof is by induction on n. Let k be a fixed nonnegative integer.
If $n = 1$, then the equation

$1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} = \int_0^n \left[(k+1)\cdot p_k(x) + C\right] dx$

reduces to $1 = 1$, since $(k+1)\int_0^1 p_k(x)\ dx = 1 - C$. Now, suppose that for any positive integer n, we have

$1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} = \int_0^n \left[(k+1)\cdot p_k(x) + C\right] dx$. Then, observe that

$$1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} + (n+1)^{k+1}$$

$$= \int_0^n \left[(k+1)\cdot p_k(x) + C\right] dx + (k+1)\int_n^{n+1} p_k(x)\ dx + C \quad \text{(by the Lemma)}$$

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$$= \ (k+1)\int_0^{n} p_k(x)\ dx \ + \ Cn \ + \ (k+1)\int_n^{n+1} p_k(x)\ dx \ + \ C$$

$$= \ (k+1)\int_0^{n+1} p_k(x)\ dx \ + \ Cn \ + \ C$$

$$= \ \int_0^{n+1} \left[(k+1)\cdot p_k(x) \ + \ C\right] dx \ .$$

3. CONCLUSION

We conclude this article with some examples to demonstrate how the algorithm is applied.

Example 1: Let's derive the right-hand side of the formula for

$1^1 + 2^1 + 3^1 + \ldots + n^1$ from the formula $1^0 + 2^0 + 3^0 + \ldots + n^0 = n$. Since $k = 0$ and the right-hand side of the above equation is n, we have $p_0(n) = n$, so $p_0(x) = x$.

First, we integrate $(0 + 1)\cdot x + C$ with respect to x from $x = 0$ to $x = n$:

$$\int_0^n (1\cdot x + C)\ dx \ = \ \left(\frac{x^2}{2} + Cx\right)\Bigg|_0^n \ = \ \frac{n^2}{2} + Cn \ .$$

Then, to find the value of the constant C, we substitute 1 for n in the equation

$1^1 + 2^1 + 3^1 + \ldots + n^1 = \dfrac{n^2}{2} + Cn$ to obtain $1 = \dfrac{1^2}{2} + C\cdot 1$, so

$C = \dfrac{1}{2}$. Hence, $1^1 + 2^1 + 3^1 + \ldots + n^1 = \dfrac{n^2 + n}{2}$.

Incidentally, since C is a constant and the equation

$1^1 + 2^1 + 3^1 + \ldots + n^1 = \dfrac{n^2}{2} + Cn$ holds for all positive integer values of n, we can substitute *any* positive integer value for n in this equation to solve for C, not just 1.

Example 2: Now, let's derive the right-hand side of the formula for

$1^2 + 2^2 + 3^2 + \ldots + n^2$ from the formula $1^1 + 2^1 + 3^1 + \ldots + n^1 = \dfrac{n^2 + n}{2}$.

Since $k = 1$ and the right-hand side of the above equation is $\dfrac{n^2 + n}{2}$, we have

$p_1(n) = \dfrac{n^2 + n}{2}$, so $p_1(x) = \dfrac{x^2 + x}{2}$. We first integrate $(1 + 1)\cdot\dfrac{x^2 + x}{2} + C$ with respect to x from $x = 0$ to $x = n$:

$$\int_0^n \left[2\cdot\left(\frac{x^2 + x}{2}\right) + C\right] dx \ = \ \left(\frac{x^3}{3} + \frac{x^2}{2} + Cx\right)\Bigg|_0^n \ = \ \frac{n^3}{3} + \frac{n^2}{2} + Cn \ .$$

Then, we evaluate C by substituting 1 for n in the equation

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n^3}{3} + \dfrac{n^2}{2} + Cn$, yielding

$1 = \dfrac{1^3}{3} + \dfrac{1^2}{2} + C \cdot 1$, so $C = \dfrac{1}{6}$.

Consequently, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n^3}{3} + \dfrac{n^2}{2} + \dfrac{n}{6} = \dfrac{2n^3 + 3n^2 + n}{6}$.

The reader is encouraged to verify the veracity of the recursive algorithm presented in this article for larger values of k. The formulas for the sum of the k^{th} powers of the first n positive integers, for each $k = 1, 2, \ldots, 10$, are listed at [7].

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